Teacher notes

The power of dimensional analysis

We all know that the period of a mass-spring system is given by $T=2\pi\sqrt{\frac{m}{k}}$. Where does this come from? We must solve the differential equation coming from Newton's second law applied to the mass-spring system, i.e. $m\frac{d^2x}{dt^2}=-kx$. But there is another way. What could the period of oscillation possibly depend on? The mass and the spring constant come to mind immediately but also the amplitude *A* i.e. the maximum displacement by which the mass is pulled to the side before being released. In other words we must have a relation of the form $T=f(m, k, A)$. For such a relation the units on the left side of the relation must match the units on the right. Assuming a multiplicative relation between the variables on the right i.e. $f(m, k, A) \sim m^{\alpha}k^{\beta}A^{\gamma}$ we must have that

$$T=cm^{\alpha}k^{\beta}A^{\gamma}$$

where *c* is a dimensionless constant. Let's now match the units on both sides of this equation:

The unit of *T* is the second: s.

The unit of the mass *m* is the kg.

The unit of *k* is N m^{-1} which is equivalent to kg m s^{-2} m^{-1} or kg s^{-2}.

The unit of *A* is the meter, m.

Then (*c* is dimensionless so it does not enter in what follows)

$\mathrm{s}=\mathrm{kg}^{\alpha}\left(\mathrm{kg\ s}^{-2}\right)^{\beta}\mathrm{m}^{\gamma}$

This simplifies to

$\mathrm{s}=\mathrm{kg}^{\alpha+\beta}\ \mathrm{s}^{-2\beta}\ \mathrm{m}^{\gamma}$

Right away we find that

$\alpha+\beta=0, \quad -2\beta=1\Rightarrow\beta=-\frac{1}{2}$ and $\gamma=0$. This means that the solution is $\alpha=\frac{1}{2}, \quad \beta=-\frac{1}{2}$ and $\gamma=0$. Hence the relation foe period becomes

$$T=c\sqrt{\frac{m}{k}}$$

The dimensionless constant *c* cannot be determined in this way. Even though we do not know *c* we have achieved a lot: for one thing, the period is independent of the amplitude *A* which was by no means obvious at the beginning. And we know how the period depends on *m* and *k*!

Now let's look at a less trivial example: consider a mass-spring system again but this time the restoring force is $-kx^3$. What can we say about the period now? Working as before we have to realize that now the units of *k* are N m^{-3} i.e. kg m s^{-2} m^{-3} i.e. kg m^{-2} s^{-2}. Then

$$\mathrm{s} = \mathrm{kg}^{\alpha}\left(\mathrm{kg\ m^{-2}\ s^{-2}}\right)^{\beta}\mathrm{m}^{\gamma}$$

i.e.

$$\mathrm{s} = \mathrm{kg}^{\alpha+\beta}\ \mathrm{m}^{-2\beta+\gamma}\ \ \mathrm{s}^{-2\beta}$$

The solution now is

$\alpha+\beta=0$ $\quad -2\beta+\gamma=0$ $\quad$ and $-2\beta=1 \Rightarrow \beta=-\dfrac{1}{2}$, i.e. $\alpha=\dfrac{1}{2},\ \ \beta=-\dfrac{1}{2}$ and $\gamma=-1$. Hence the relation for the period becomes $T=\dfrac{c}{A}\sqrt{\dfrac{m}{k}}$. It would take more serious work (see appendix) to solve the differential equation for this problem, $m\dfrac{d^2x}{dt^2}=-kx^3$, compared to the harmonic oscillator $m\dfrac{d^2x}{dt^2}=-kx$. Yet, by the method of dimensional analysis we have discovered the essential features of this motion. We see the non-intuitive result that as the amplitude increases, the period decreases! The period is independent of the amplitude only for the case of the harmonic oscillator.

We finally turn to a much less trivial and much more important example. We know that black bodies at some temperature *T* radiate EM waves. We also know that the details of the body itself are irrelevant: the total intensity (power per unit area) radiated only depends on temperature. We also know that different amounts of intensity are emitted at different wavelengths. The term for this is spectral intensity B_λ: the quantity $B_\lambda d\lambda$ is the intensity radiated at wavelengths in the interval $(\lambda,\ \lambda+d\lambda)$. The total intensity radiated is then $\int_0^\infty B_\lambda d\lambda$. What does dimensional analysis say for B_λ?

We know that B_λ cannot depend on the details of the body. It has to depend on temperature *T*. We are emitting EM waves so B_λ could also depend on the speed of light *c*. Different intensities are radiated at different wavelengths so B_λ has to depend on the wavelength λ as well. But we are dealing with thermal physics and there is one constant that is relevant and could appear in B_λ, namely the Boltzmann constant *k*. So, we assume that

$$B_\lambda = f(c,\ \lambda, T, k)$$

The units of B_λ are W m^{-2} m^{-1} i.e. kg m^{-1} s^{-3}. So we have

$$\text{kg m}^{-1}\text{ s}^{-3} = c^\alpha \lambda^\beta T^\gamma k^\delta$$

or

$$\text{kg m}^{-1}\text{ s}^{-3} = \left(\text{m s}^{-1}\right)^\alpha \text{m}^\beta \text{K}^\gamma \left(\text{kg m}^2\text{ s}^{-2}\text{ K}^{-1}\right)^\delta$$

which results in

$$\text{kg m}^{-1}\text{ s}^{-3} = \text{m}^{\alpha+\beta+2\delta}\ \text{K}^{\gamma-\delta}\ \text{kg}^\delta\ \text{s}^{-\alpha-2\delta}$$

The solution is $\gamma - \delta = 0$, $\delta = 1$, $\alpha + \beta + 2\delta = -1$ and $-\alpha - 2\delta = -3$. In other words,

$\alpha = 1,\ \beta = -4,\ \gamma = 1 \text{ and } \delta = 1$ resulting in $B_\lambda \sim \frac{ckT}{\lambda^4}$. Rayleigh and Jeans derived this analytically at the end of the nineteenth century and got the overall dimensionless constant which our dimensional analysis cannot give. It was 2, so $B_\lambda = \frac{2ckT}{\lambda^4}$.

This is a famous result. It says that more and more intensity is radiated at smaller and smaller wavelengths (resulting overall in an infinite amount of energy radiated by the black body). This is known as the ultra-violet catastrophe and signaled the end of classical physics. We know that Planck solved this problem with the introduction of what later were called photons, quanta of the EM field, each carrying energy hf or $\frac{hc}{\lambda}$ and that the correct B_λ is given by $B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$.

What is extraordinary here is that dimensional analysis performed *within classical physics* points to the demise of classical physics.

Appendix

We need to solve $m\frac{d^2x}{dt^2} = -kx^3$, i.e. $\frac{d^2x}{dt^2} = -\frac{k}{m}x^3$. The acceleration can be rewritten as $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{1}{2}\frac{d}{dx}v^2$ and so

$$\frac{1}{2}\frac{d}{dx}v^2 = -\frac{k}{m}x^3$$

Integrating once we find

$$\frac{1}{2}v^2 = C - \frac{k}{4m}x^4$$

When *x* = *A* (the amplitude), *v* = 0 so $C = \frac{k}{4m}A^4$ and hence

$$v = \pm\sqrt{\frac{k}{2m}A^4 - \frac{k}{2m}x^4}$$

Finally,

$\frac{dx}{dt} = \pm\sqrt{\frac{k}{2m}A^4 - \frac{k}{2m}x^4}$ so that $dt = \pm\frac{dx}{\sqrt{\frac{k}{2m}A^4 - \frac{k}{2m}x^4}} = \pm\sqrt{\frac{2m}{k}}\frac{dx}{\sqrt{A^4 - x^4}}$.

$T = 4\sqrt{\frac{2m}{k}}\int_0^A \frac{dx}{\sqrt{A^4 - x^4}} = 4\sqrt{\frac{2m}{k}}\int_0^A \frac{dx}{A^2\sqrt{1 - \frac{x^4}{A^4}}}$. Change variables to $u = \frac{x}{A}$ so that (with *A du* = *dx*)

$$T = \frac{4}{A}\sqrt{\frac{2m}{k}}\int_0^1 \frac{du}{\sqrt{1 - u^4}}.$$

The integral is just a numerical factor (approx. 1.311) and we have agreement with the result from dimensional analysis, $T \sim \frac{1}{A}\sqrt{\frac{m}{k}}$.

Incidentally, this shows that if the restoring force law is a power, $m\frac{d^2x}{dt^2} = -kx^n$ (*n* is a positive number) we can work as before to get

$$\frac{1}{2}\frac{d}{dx}v^2 = -\frac{k}{m}x^n$$

$$\frac{1}{2}v^2 = C - \frac{k}{m(n+1)}x^{n+1}$$

$$v = \pm\sqrt{\frac{2k}{m(n+1)}A^{n+1} - \frac{2k}{m(n+1)}x^{n+1}}$$

$T = 4\sqrt{\frac{m(n+1)}{2k}}\int_0^A \frac{dx}{\sqrt{A^{n+1} - x^{n+1}}} = 4\sqrt{\frac{m(n+1)}{2k}}\int_0^A \frac{dx}{A^{\frac{n+1}{2}}\sqrt{1 - \frac{x^{n+1}}{A^{n+1}}}}$. Finally,

$$T = \frac{4}{A^{\frac{n-1}{2}}}\sqrt{\frac{m(n+1)}{2k}}\int_0^1 \frac{du}{\sqrt{1-u^{n+1}}}$$

Again, we see that $T \sim \frac{1}{A^{\frac{n-1}{2}}}\sqrt{\frac{m}{k}}$. The period is independent of the amplitude **only** for $n = 1$.